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Boundary Reflection Matrix for *ade* Affine Toda Field Theory

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ABSTRACT

We present a complete set of conjectures for the exact boundary reflection matrix for *ade* affine Toda field theory defined on a half line with the Neumann boundary condition.

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I. Introduction

About a decade ago, studies on the integrable quantum field theory defined on a half line ($-\infty < x \leq 0$) was initiated using symmetry principles under the assumption that the quantum integrability of the model remains intact[1]. The boundary Yang-Baxter equation, unitarity relation for boundary reflection matrix $K_a^b(\theta)$ which is conceived to describe the scattering process off a wall was introduced[1].

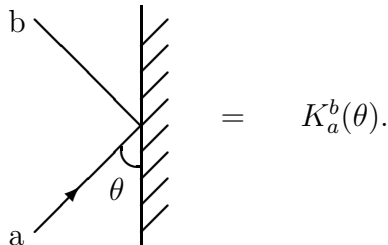


Figure 1. Boundary Reflection Matrix.

Recently, the boundary crossing-unitarity relations[2] and the boundary bootstrap equations[3] have been introduced. Subsequently, a variety of solutions of the algebraic equations for affine Toda field theory has been constructed[2, 3, 4, 5, 6, 7]. However, a proper interpretation of these solutions in terms of the Lagrangian quantum field theory had been unknown.

On the other hand, nontrivial boundary potentials which do not destroy the integrability properties in the sense that there still exists infinite number of conserved currents has been determined[2, 6, 7, 8, 9, 10]. The stability problem of a certain models with boundary potential has been also discussed[7, 11].

Very recently, we have proposed a formalism[12] to compute a boundary reflection matrix in the framework of the Lagrangian quantum field theory with a boundary[13, 14, 15]. The idea is to extract the boundary reflection matrix directly from the two-point correlation function in the coordinate space. And it has revealed a number of striking features of the perturbative quantum field theory defined on a half line.

Using this formalism, we determined the exact boundary reflection matrix for sinh-Gordon model($a_1^{(1)}$ affine Toda theory) and Bullough-Dodd model($a_2^{(2)}$ affine Toda theory) with the Neumann boundary condition modulo a ‘universal mysteri-

ous factor half'[12]. If we assume the strong-weak coupling 'duality', these solutions are unique.

Above two models have a particle spectrum with only one mass. On the other hand, when the theory has a particle spectrum with more than one mass, each one loop contribution from different types of Feynman diagrams has a variety of non-meromorphic terms. We expect actual cancellation of these non-meromorphic terms ought to be essential for a boundary reflection matrix to have a nice analytic property.

In Ref.[16], we evaluated one loop boundary reflection matrix for $d_4^{(1)}$ affine Toda field theory and showed a remarkable cancellation of non-meromorphic terms among themselves. This result also enabled us to determine the exact boundary reflection matrix uniquely under the assumption of the strong-weak coupling 'duality'. It turned out that the boundary reflection matrix has singularities which can be accounted for by a new type of singularities of Feynman diagrams for a theory defined on a half line.

In this paper, we present a complete set of conjectures for the exact boundary reflection matrix for *ade* affine Toda field theory defined on a half line with the Neumann boundary condition. With this boundary condition, we expect the strong-weak coupling 'duality' which is a symmetry of the model defined on a full line is still effective.

In section II, we review the formalism developed in Ref.[12]. Especially, we give a more informative form of the formulae given in Ref.[12]. In section III, we present a complete set of conjectures for the exact boundary reflection matrix for *ade* affine Toda field theory with the Neumann boundary condition. Finally, we make conclusions in section IV. In appendix, we present one loop result as well as the complete set of solutions of the boundary bootstrap equations for $a_3^{(1)}$ theory.

II. Boundary Reflection Matrix

The action for affine Toda field theory defined on a half line ($-\infty < x \leq 0$) is given by

$$S(\Phi) = \int_{-\infty}^0 dx \int_{-\infty}^{\infty} dt \left(\frac{1}{2} \partial_{\mu} \phi^a \partial^{\mu} \phi^a - \frac{m^2}{\beta^2} \sum_{i=0}^r n_i e^{\beta \alpha_i \cdot \Phi} \right), \quad (1)$$

where

$$\alpha_0 = - \sum_{i=1}^r n_i \alpha_i, \quad \text{and} \quad n_0 = 1.$$

The field ϕ^a ($a = 1, \dots, r$) is a -th component of the scalar field Φ , and α_i ($i = 1, \dots, r$) are simple roots of a Lie algebra g with rank r normalized so that the universal function $B(\beta)$ through which the dimensionless coupling constant β appears in the S -matrix takes the following form:

$$B(\beta) = \frac{1}{2\pi} \frac{\beta^2}{(1 + \beta^2/4\pi)}. \quad (2)$$

The m sets the mass scale and the n_i s are the so-called Kac labels which are characteristic integers defined for each Lie algebra.

Here we consider the model with no boundary potential, which corresponds to the Neumann boundary condition: $\frac{\partial \phi^a}{\partial x} = 0$ at $x = 0$. This case is believed to be quantum stable in the sense that the existence of a boundary does not change the structure of the quantum spectrum determined for the same theory defined on a full line.

In classical field theory, it is quite clear how we extract the boundary reflection matrix. It is the coefficient of reflection term in the classical two-point correlation function namely it is 1:

$$\begin{aligned} G_N(t', x'; t, x) &= G(t', x'; t, x) + G(t', x'; t, -x) \\ &= \int \frac{d^2 p}{(2\pi)^2} \frac{i}{p^2 - m_a^2 + i\varepsilon} e^{-iw(t'-t)} (e^{ik(x'-x)} + e^{ik(x'+x)}). \end{aligned} \quad (3)$$

We may use the k -integrated version:

$$G_N(t', x'; t, x) = \int \frac{dw}{2\pi} \frac{1}{2\bar{k}} e^{-iw(t'-t)} (e^{i\bar{k}|x'-x|} + e^{-i\bar{k}(x'+x)}), \quad \bar{k} = \sqrt{w^2 - m_a^2}. \quad (4)$$

We find that the unintegrated version is very useful to extract the asymptotic part of the two-point correlation function far away from the boundary.

In quantum field theory, it also seems quite natural to extend above idea in order to extract the quantum boundary reflection matrix directly from the quantum two-point correlation function. This idea has been pursued in Ref.[12] to extract one loop boundary reflection matrix.

To compute two-point correlation functions at one loop order, we follow the idea of the conventional perturbation theory[13, 14, 15]. That is, we generate relevant Feynman diagrams and then evaluate each of them by using the zero-th order two-point function for each line occurring in the Feynman diagrams.

At one loop order, there are three types of Feynman diagram contributing to the two-point correlation function as depicted in figure 2.

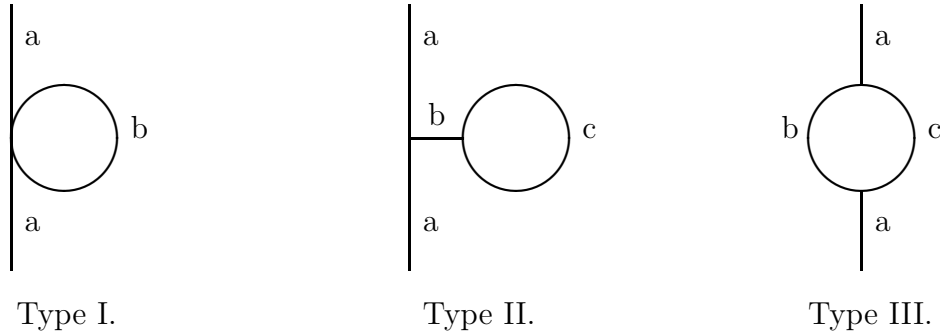


Figure 2. Diagrams for the one loop two-point function.

For a theory defined on a full line which has translational symmetry in space and time direction, Type I, II diagrams have logarithmic infinity independent of the external energy-momenta and are the only divergent diagrams in 1+1 dimensions. This infinity is usually absorbed into the infinite mass renormalization. Type III diagrams have finite corrections depending on the external energy-momenta and produces a double pole to the two-point correlation function.

The remedy for these double poles is to introduce a counter term to the original Lagrangian to cancel this term(or to renormalize the mass). In addition, to maintain the residue of the pole, we have to introduce wave function renormalization. Then the renormalized two-point correlation function remains the same as the tree level

one with renormalized mass m_a , whose ratios are the same as the classical value. This mass renormalization procedure can be generalized to arbitrary order of loops.

Now let us consider each diagram for a theory defined on a half line. Type I diagram gives the following contribution:

$$\int_{-\infty}^0 dx_1 \int_{-\infty}^{\infty} dt_1 G_N(t, x; t_1, x_1) G_N(t', x'; t_1, x_1) G_N(t_1, x_1; t_1, x_1). \quad (5)$$

From Type II diagram, we can read off the following expression:

$$\int_{-\infty}^0 dx_1 dx_2 \int_{-\infty}^{\infty} dt_1 dt_2 G_N(t, x; t_1, x_1) G_N(t', x'; t_1, x_1) G_N(t_1, x_1; t_2, x_2) G_N(t_2, x_2; t_2, x_2). \quad (6)$$

Type III diagram gives the following contribution:

$$\int_{-\infty}^0 dx_1 dx_2 \int_{-\infty}^{\infty} dt_1 dt_2 G_N(t, x; t_1, x_1) G_N(t', x'; t_2, x_2) G_N(t_2, x_2; t_1, x_1) G_N(t_2, x_2; t_1, x_1). \quad (7)$$

After the infinite as well as finite mass renormalization, the remaining terms coming from type I, II and III diagrams can be written as follows with different I_i functions[12]:

$$\int \frac{dw}{2\pi} \frac{dk}{2\pi} \frac{dk'}{2\pi} e^{-iw(t'-t)} e^{i(kx+k'x')} \frac{i}{w^2 - k^2 - m_a^2 + i\varepsilon} \frac{i}{w^2 - k'^2 - m_a^2 + i\varepsilon} I_i(w, k, k'). \quad (8)$$

Contrary to the other terms which resemble those of a full line, this integral has two spatial momentum integration.

In the asymptotic region far away from the boundary, these terms can be evaluated up to exponentially damped term as x, x' go to $-\infty$, yielding the following result for the elastic boundary reflection matrix $K_a(\theta)$ defined as the coefficient of the reflected term of the two-point correlation function:

$$\int \frac{dw}{2\pi} e^{-iw(t'-t)} \frac{1}{2k} (e^{i\bar{k}|x'-x|} + K_a(w) e^{-i\bar{k}(x'+x)}), \quad \bar{k} = \sqrt{w^2 - m_a^2}. \quad (9)$$

$K_a(\theta)$ is obtained using $w = m_a c h \theta$. Here we list each one loop contribution to $K_a(\theta)$ from the three types of diagram depicted in figure 2[12]:

$$K_a^{(I)}(\theta) = \frac{1}{4m_a s h \theta} \left(\frac{1}{2\sqrt{m_a^2 s h^2 \theta + m_b^2}} + \frac{1}{2m_b} \right) C_1 S_1, \quad (10)$$

$$K_a^{(II)}(\theta) = \frac{1}{4m_a sh\theta} \left(\frac{-i}{(4m_a^2 sh^2\theta + m_b^2)2\sqrt{m_a^2 sh^2\theta + m_c^2}} + \frac{-i}{2m_b^2 m_c} \right) C_2 S_2, \quad (11)$$

$$K_a^{(III)}(\theta) = \frac{1}{4m_a sh\theta} (4I_3(k_1 = 0, k_2 = \bar{k}) + 4I_3(k_1 = \bar{k}, k_2 = 0)) C_3 S_3, \quad (12)$$

where a ‘universal mysterious factor half’ is included. C_i, S_i denote numerical coupling factors and symmetry factors, respectively. I_3 is defined by

$$I_3 \equiv \frac{1}{4} \left(\frac{i}{2\bar{w}_1(\bar{w}_1 - \tilde{w}_1^+)(\bar{w}_1 - \tilde{w}_1^-)} + \frac{i}{(\tilde{w}_1^+ - \bar{w}_1)(\tilde{w}_1^+ + \bar{w}_1)(\tilde{w}_1^+ - \tilde{w}_1^-)} \right), \quad (13)$$

where

$$\bar{w}_1 = \sqrt{k_1^2 + m_b^2}, \quad \tilde{w}_1^+ = w + \sqrt{k_2^2 + m_c^2}, \quad \tilde{w}_1^- = w - \sqrt{k_2^2 + m_c^2}. \quad (14)$$

It should be remarked that this term should be symmetrized with respect to m_b, m_c with a half.

The expression for a contribution from Type III diagram can be rewritten in the following form:

$$\begin{aligned} K_a^{(III)} &= \frac{i}{4m_a sh\theta} C_3 S_3 \\ &\left(\frac{\cos \theta_{ab}^c}{4m_a m_b^2 (ch^2\theta - \cos^2 \theta_{ab}^c)} - \frac{m_a ch^2\theta + m_b \cos \theta_{ab}^c}{2m_a m_b^2 2\sqrt{m_a^2 sh^2\theta + m_c^2} (ch^2\theta - \cos^2 \theta_{ab}^c)} \right. \\ &\left. + \frac{\cos \theta_{ac}^b}{4m_a m_c^2 (ch^2\theta - \cos^2 \theta_{ac}^b)} - \frac{m_a ch^2\theta + m_c \cos \theta_{ac}^b}{2m_a m_c^2 2\sqrt{m_a^2 sh^2\theta + m_b^2} (ch^2\theta - \cos^2 \theta_{ac}^b)} \right), \end{aligned} \quad (15)$$

where θ_{ab}^c is a usual fusion angle defined by

$$\cos \theta_{ab}^c = \frac{m_c^2 - m_a^2 - m_b^2}{2m_a m_b}. \quad (16)$$

Let us note a few interesting points. Firstly, all the expressions in Eqs.(10,11,12) have in general non-meromorphic terms when the theory has a mass spectrum with more than one mass. Cancellation of these terms is expected to occur for the boundary reflection matrix to have a nice analytic property. We have verified this non-trivial cancellation for $d_4^{(1)}$ theory in Ref.[16] and the result for $a_3^{(1)}$ theory is presented in this appendix. Secondly, the Feynman diagrams have (simple pole)singularities

which are absent for the theory defined on a full line. A general study on the analytic property of the boundary reflection matrix is definitely needed, while that for the scattering matrix has been extensively done[17].

Moreover, the position of poles are directly related with fusion angles as in Eq.(15) and less obviously as in Eq.(11). Later in the appendix, we will see a nontrivial cancellation of non-meromorphic terms and the fact that the new type of singularities accounts for the singularities of the exact boundary reflection matrix.

III. The Boundary Reflection Matrix for *ade* affine Toda theory

The exact S -matrix for integrable quantum field theory defined on a full line has been conjectured using the symmetry principles such as Yang-Baxter equation, unitarity, crossing relation, real analyticity and bootstrap equation[18, 19, 20, 21, 22]. This program entirely relies on the assumed quantum integrability of the model as well as the fundamental assumptions such as strong-weak coupling ‘duality’ and ‘minimality’.

In order to determine the exact S -matrix uniquely, Feynman’s perturbation theory has been used[23, 24, 25, 26, 27] and shown to agree well with the conjectured ‘minimal’ S -matrices. In perturbation theory, S -matrix is extracted from the four-point correlation function with LSZ reduction formalism. Especially, the singularity structures were examined in terms of Landau singularity[17], of which odd order poles are interpreted as coming from the intermediate bound states.

In determining the whole set of scattering matrix elements, it is essentially sufficient to determine the element for the so-called ‘elementary particle’. Starting from that element, we can determine all the other elements using the bootstrap equations[20]. This is also true for the boundary reflection matrix. In $a_n^{(1)}$ theory, ‘elementary particle’ is the lightest one corresponding to two end points of the Dynkin digram. In $d_n^{(1)}$ theory, ‘elementary particles’ are those corresponding to (anti-)spinor representations. In $e_6^{(1)}$ theory, ‘elementary particles’ are the lightest ones which are

conjugate to each other corresponding to two end points of the Dynkin diagram. In $e_7^{(1)}$ and $e_8^{(1)}$ theories, it is the lightest one corresponding to the end point of the longer arm of the Dynkin diagram.



Figure 3. Dynkin diagram for a_n .

Let us start from $a_n^{(1)} (n \geq 1)$ theory. The boundary reflection matrix for the ‘elementary particles’ can be coded into the following pyramid of exponents of the factors $[x]$ which appear in the boundary reflection matrix.

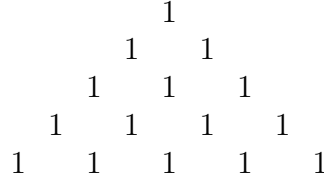


Figure 4. A pyramid of exponents for a_n theory.

It means

$$K_1(\theta) = K_n(\theta) = \prod_{k=1, \text{step } 2}^{2h-3} [k/2], \quad (17)$$

where

$$[x] = \frac{(x - 1/2)(x + 1/2)}{(x - 1/2 + B/2)(x + 1/2 - B/2)}, \quad (x) = \frac{sh(\theta/2 + i\pi x/2h)}{sh(\theta/2 - i\pi x/2h)}. \quad (18)$$

From these elements of the boundary reflection matrix, we can in principle determine all the other elements using the boundary bootstrap equations.

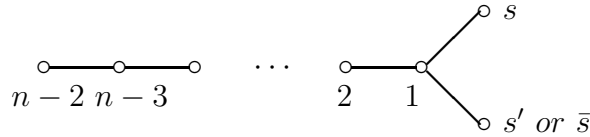


Figure 5. Dynkin diagram for d_n .

For $d_n^{(1)} (n \geq 2)$ theory, a pyramid of exponents takes a slightly complicated form. $d_2^{(1)}$ theory is equal to two copies of sinh-Gordon theory which is $a_1^{(1)}$ theory and $d_3^{(1)}$

theory is equal to $a_3^{(1)}$ theory.

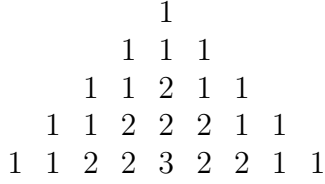


Figure 6. The first pyramid of exponents for d_n theory.

It means

$$K_s(\theta) = K_{s'(\bar{s})}(\theta) = \prod_{k=1, \text{step} 2}^{2h-3} [k/2]^{x_k}, \quad (19)$$

where x_k are the exponents in sequence from(to) left to(from) right in figure 6. The rule of the figure 6 is the following. At odd rows except the apex, prepare two copies of the middle number and put them to two sites neighbouring to the center, pushing the others away towards both sides and increment the original middle number by one unit. At even rows, do the same thing as for odd rows but leave the middle number without incrementing. From these elements of the boundary reflection matrix, we can determine all the other elements.

On the other hand, a pyramid of exponents for lightest particle corresponding to the end point of the longer arm of the Dynkin diagram take the following form.

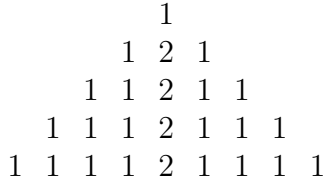


Figure 7. The second pyramid of exponents for d_n theory.

It means

$$K_{n-2}(\theta) = \prod_{k=1, \text{step} 2}^{2h-3} [k/2]^{x_k}, \quad (20)$$

where x_k are the exponents in figure 7. The rule of the figure 7 is that only the middle number is two except the apex. From these data, we can not determine all the other elements for each $d_n^{(1)}$ theory. However, it obviously looks simpler than the

elements corresponding to (anti-)spinor representations.

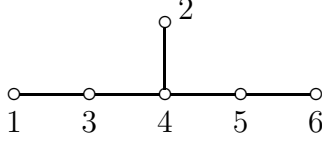


Figure 8. Dynkin diagram for e_6 .

For $e_n^{(1)}$ theory, we have checked the conjectured boundary reflection matrices of the ‘elementary particles’ by perturbation theory. Other elements for particles which are not ‘elementary’ are determined using the boundary bootstrap equations.

For $e_6^{(1)}$ theory ($h = 12$), a complete list is

$$K_1(\theta) = [1/2][3/2][5/2][7/2]^2[9/2]^2[11/2]^2[13/2]^2[15/2]^2[17/2][19/2][21/2], \quad (21)$$

$$K_2(\theta) = [1/2][3/2][5/2]^2[7/2]^3[9/2]^3[11/2]^3[13/2]^2[15/2]^3[17/2]^2[19/2][21/2],$$

$$K_3(\theta) = [1/2][3/2]^2[5/2]^3[7/2]^4[9/2]^4[11/2]^4[13/2]^4[15/2]^3[17/2]^2[19/2]^2[21/2],$$

$$K_4(\theta) = [1/2][3/2]^3[5/2]^5[7/2]^6[9/2]^6[11/2]^6[13/2]^5[15/2]^4[17/2]^3[19/2]^2[21/2],$$

$$K_5(\theta) = K_3(\theta),$$

$$K_6(\theta) = K_1(\theta).$$

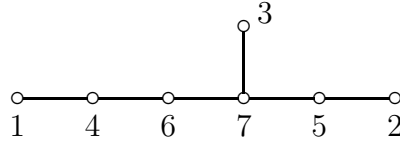


Figure 9. Dynkin diagram for e_7 .

For $e_7^{(1)}$ theory ($h = 18$), we report only two elements for technical reasons. An interested reader should find no difficulty in producing all the other elements. A partial list is

$$K_1(\theta) = [1/2][3/2][5/2][7/2][9/2]^2[11/2]^2[13/2]^2[15/2]^2[17/2]^3[19/2]^2 \quad (22)$$

$$[21/2]^2[23/2]^2[25/2]^2[27/2][29/2][31/2][33/2],$$

$$K_2(\theta) = [1/2][3/2][5/2][7/2]^2[9/2]^2[11/2]^3[13/2]^3[15/2]^3[17/2]^3[19/2]^2$$

$$[21/2]^3[23/2]^3[25/2]^2[27/2]^2[29/2][31/2][33/2].$$

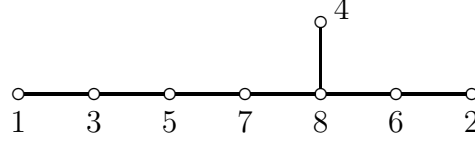


Figure 10. Dynkin diagram for e_8 .

For $e_8^{(1)}$ theory ($h = 30$), a complete list is

$$\begin{aligned}
K_1 &= [1/2][3/2][5/2][7/2][9/2][11/2]^2[13/2]^2[15/2]^2[17/2]^2[19/2]^3 \\
&\quad [21/2]^3[23/2]^3[25/2]^3[27/2]^3[29/2]^4[31/2]^3[33/2]^3[35/2]^3[37/2]^3[39/2]^3 \\
&\quad [41/2]^2[43/2]^2[45/2]^2[47/2]^2[49/2][51/2][53/2][55/2][57/2], \\
K_2 &= [1/2][3/2][5/2][7/2]^2[9/2]^2[11/2]^3[13/2]^4[15/2]^4[17/2]^5[19/2]^5 \\
&\quad [21/2]^4[23/2]^5[25/2]^5[27/2]^5[29/2]^5[31/2]^4[33/2]^5[35/2]^5[37/2]^4[39/2]^4 \\
&\quad [41/2]^4[43/2]^4[45/2]^4[47/2]^3[49/2]^2[51/2]^2[53/2][55/2][57/2], \\
K_3 &= [1/2][3/2]^2[5/2]^2[7/2]^2[9/2]^3[11/2]^4[13/2]^4[15/2]^4[17/2]^5[19/2]^6 \\
&\quad [21/2]^6[23/2]^6[25/2]^6[27/2]^7[29/2]^7[31/2]^6[33/2]^6[35/2]^6[37/2]^6[39/2]^5 \\
&\quad [41/2]^4[43/2]^4[45/2]^4[47/2]^3[49/2]^2[51/2]^2[53/2]^2[55/2]^2[57/2], \\
K_4 &= [1/2][3/2]^2[5/2]^3[7/2]^3[9/2]^4[11/2]^5[13/2]^5[15/2]^6[17/2]^6[19/2]^7 \\
&\quad [21/2]^7[23/2]^7[25/2]^7[27/2]^8[29/2]^9[31/2]^8[33/2]^8[35/2]^6[37/2]^6[39/2]^6 \\
&\quad [41/2]^5[43/2]^5[45/2]^4[47/2]^4[49/2]^3[51/2]^3[53/2]^2[55/2]^2[57/2]^2, \\
K_5 &= [1/2][3/2]^2[5/2]^3[7/2]^4[9/2]^6[11/2]^7[13/2]^6[15/2]^7[17/2]^8[19/2]^9 \\
&\quad [21/2]^9[23/2]^9[25/2]^{10}[27/2]^{10}[29/2]^9[31/2]^8[33/2]^9[35/2]^9[37/2]^8[39/2]^7 \\
&\quad [41/2]^6[43/2]^6[45/2]^5[47/2]^4[49/2]^4[51/2]^4[53/2]^3[55/2]^2[57/2], \\
K_6 &= [1/2][3/2]^2[5/2]^3[7/2]^4[9/2]^5[11/2]^7[13/2]^8[15/2]^9[17/2]^{10}[19/2]^{10} \\
&\quad [21/2]^{10}[23/2]^{10}[25/2]^{10}[27/2]^{10}[29/2]^{10}[31/2]^9[33/2]^9[35/2]^9[37/2]^8[39/2]^8 \\
&\quad [41/2]^7[43/2]^7[45/2]^7[47/2]^5[49/2]^4[51/2]^3[53/2]^2[55/2]^2[57/2], \\
K_7 &= [1/2][3/2]^2[5/2]^4[7/2]^6[9/2]^7[11/2]^8[13/2]^9[15/2]^{10}[17/2]^{11}[19/2]^{12} \\
&\quad [21/2]^{12}[23/2]^{13}[25/2]^{13}[27/2]^{13}[29/2]^{13}[31/2]^{12}[33/2]^{12}[35/2]^{11}[37/2]^{10}[39/2]^9 \\
&\quad [41/2]^8[43/2]^7[45/2]^6[47/2]^5[49/2]^4[51/2]^4[53/2]^3[55/2]^2[57/2], \\
K_8 &= [1/2]^2[3/2]^4[5/2]^6[7/2]^8[9/2]^9[11/2]^{11}[13/2]^{12}[15/2]^{13}[17/2]^{14}[19/2]^{15}
\end{aligned} \tag{23}$$

$$[21/2]^{15}[23/2]^{15}[25/2]^{16}[27/2]^{16}[29/2]^{17}[31/2]^{16}[33/2]^{14}[35/2]^{13}[37/2]^{11}[39/2]^{10} \\ [41/2]^9[43/2]^8[45/2]^7[47/2]^6[49/2]^5[51/2]^4[53/2]^4[55/2]^3[57/2]^2[59/2].$$

We remark that we have extensive direct proofs for these conjectures by perturbation theory which are basically case-by-case works. Parts of them have been already presented in Refs.[12, 16] and are presented in the appendix of this paper. These conjectured boundary reflection matrices are also tested against various algebraic requirements such as the boundary crossing-unitarity relations and it always gives consistent results.

IV. Conclusions

In this paper, we presented a complete set of conjectures for the exact boundary reflection matrix for *ade* affine Toda field theory defined on a half line with Neumann boundary condition. These conjectures are based on extensive direct proofs by perturbation theory and are tested against various algebraic requirements such as the boundary crossing-unitarity relations and the boundary bootstrap equations.

Surprisingly enough, these solutions have very rich pole structures in physical strip($0 \leq \text{Im}(\theta) < \pi$). However, structures of these singularities are explainable in terms of Feynman diagrams in figure 2 which definitely have no singularity for the theory defined on a full line and their positions of poles which are produced by the Feynman diagrams are related with fusing angles for affine Toda field theory as in Eq.(15).

In the appendix, we presented a detailed computation for $a_3^{(1)}$ affine Toda field theory up to one loop order in order to demonstrate a remarkable cancellation of non-meromorphic terms which are always present for each diagram when the model has a particle spectrum with more than one mass. Using this result, we also determined the exact boundary reflection matrix under the assumption of the strong-weak coupling ‘duality’, which turned out to be ‘non-minimal’. We also presented the complete set of solutions of the boundary bootstrap equations.

Finally, we remark that a ‘universal mysterious factor half’ which is included in Eqs.(10,11,12) needs a proper explanation.

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A note added: The missing ‘universal mysterious factor half’ is found to be coming from the delta function integral(s) of the loop spatial momentum(a). That is, $\int dk \delta(2k) = 1/2$ instead of 1!

Appendix: $a_3^{(1)}$ affine Toda theory

We have to fix the normalization of roots so that the standard $B(\beta)$ function takes the form in Eq.(2).

We use the Lagrangian density given by

$$\begin{aligned} V(\Phi) = & 2m^2\phi_1\phi_1^* + 2m^2\phi_2\phi_2 + im^2\beta\phi_1\phi_1\phi_2 - im^2\beta\phi_2\phi_1^*\phi_1^* \\ & - \frac{1}{24}m^2\beta^2\phi_1\phi_1\phi_1\phi_1 + \frac{1}{4}m^2\beta^2\phi_1\phi_1\phi_1^*\phi_1^* + m^2\beta^2\phi_1\phi_1^*\phi_2\phi_2 \\ & + \frac{1}{6}m^2\beta^2\phi_2\phi_2\phi_2\phi_2 - \frac{1}{24}m^2\beta^2\phi_1^*\phi_1^*\phi_1^*\phi_1^* + O(\beta^3). \end{aligned} \quad (24)$$

The scattering matrix of this model is given by[19]

$$\begin{aligned} S_{11}(\theta) = S_{33}(\theta) = \{1\}, \quad S_{12}(\theta) = \{2\}, \quad S_{22}(\theta) = \{1\}\{3\}, \\ \{x\} = \frac{(x-1)(x+1)}{(x-1+B)(x+1-B)}. \end{aligned} \quad (25)$$

Here B is the same function defined in Eq.(2). For this model, $h = 4$ and from now on we set $m = 1$.

First we consider the light particle corresponding to ϕ_1 or its conjugate. It is understood that a suitable choice between a conjugate pair has to be made depending on a chosen direction of time flow. There are two possible configurations for Type I diagram. One is $b = \phi_1$ or its conjugate and the other is $b = \phi_2$ in the notation of figure 2. ϕ_1 loop contribution is the following:

$$K_1(\theta)^{(I-1)} = \frac{1}{4\sqrt{2}sh\theta} \left(\frac{1}{2\sqrt{2}ch\theta} + \frac{1}{2\sqrt{2}} \right) \times \left(\frac{-i}{4}\beta^2 \right) \times 4. \quad (26)$$

ϕ_2 loop contribution is the following:

$$K_1(\theta)^{(I-2)} = \frac{1}{4\sqrt{2}sh\theta} \left(\frac{1}{2\sqrt{2sh^2\theta+4}} + \frac{1}{4} \right) \times (-i\beta^2) \times 1. \quad (27)$$

There are no configurations for type II diagram for $a_3^{(1)}$ model. In fact, this is the case for any $a_n^{(1)}$ theory.

For type III diagram, there exists only one configuration with $b = \phi_1, c = \phi_2$ symmetrized. For $b = \phi_1, c = \phi_2$, when $k_1 = 0, k_2 = k$,

$$\bar{w}_1 = \sqrt{2}, \quad \tilde{w}_1^+ = \sqrt{2ch\theta} + \sqrt{2sh^2\theta+4}, \quad \tilde{w}_1^- = \sqrt{2ch\theta} - \sqrt{2sh^2\theta+4}, \quad (28)$$

and when $k_1 = k, k_2 = 0$,

$$\bar{w}_1 = \sqrt{2sh^2\theta + 2}, \quad \tilde{w}_1^+ = \sqrt{2}ch\theta + 2, \quad \tilde{w}_1^- = \sqrt{2}ch\theta - 2. \quad (29)$$

For $b = \phi_2, c = \phi_1$, when $k_1 = 0, k_2 = k$,

$$\bar{w}_1 = 2, \quad \tilde{w}_1^+ = \sqrt{2}ch\theta + \sqrt{2sh^2\theta + 2}, \quad \tilde{w}_1^- = \sqrt{2}ch\theta - \sqrt{2sh^2\theta + 2}, \quad (30)$$

and when $k_1 = k, k_2 = 0$,

$$\bar{w}_1 = \sqrt{2sh^2\theta + 4}, \quad \tilde{w}_1^+ = \sqrt{2}ch\theta + \sqrt{2}, \quad \tilde{w}_1^- = \sqrt{2}ch\theta - \sqrt{2}. \quad (31)$$

The result for Type III diagram can be obtained by inserting above data into Eq.(12):

$$K_1(\theta)^{(III)} = \frac{1}{4\sqrt{2}sh\theta} \left(-\frac{i}{8\sqrt{2}ch\theta} - \frac{i}{8\sqrt{2}\sqrt{sh^2\theta + 2}} + \frac{i}{16(\sqrt{2}ch\theta + 1)} \right) \times (-\beta^2) \times 4. \quad (32)$$

Adding the above contributions as well as the classical value 1, boundary reflection matrix for the light particle is given by

$$K_1(\theta) = 1 + \frac{i\beta^2}{16} \left(\frac{sh\theta}{ch\theta + 1/\sqrt{2}} - \frac{sh\theta}{ch\theta - 1} \right) + O(\beta^4). \quad (33)$$

The unwanted non-meromorphic terms exactly cancel out.

Now we consider the heavy particle corresponding to ϕ_2 which are self conjugate. There are two possible configurations for Type I diagram. One is $b = \phi_1$, the other is $b = \phi_2$ in the notation of figure 2. ϕ_2 loop contribution is the following:

$$K_2(\theta)^{(I-1)} = \frac{1}{8sh\theta} \left(\frac{1}{4ch\theta} + \frac{1}{4} \right) \times \left(\frac{-i}{6}\beta^2 \right) \times 12. \quad (34)$$

ϕ_1 loop contribution is the following:

$$K_2(\theta)^{(I-2)} = \frac{1}{8sh\theta} \left(\frac{1}{2\sqrt{4sh^2\theta + 2}} + \frac{1}{2\sqrt{2}} \right) \times (-i\beta^2) \times 2. \quad (35)$$

There is no type II diagram for the heavy particle, either.

For type III diagram, there is single configuration with $b = \phi_1, c = \phi_1$. When $k_1 = 0, k_2 = k$,

$$\bar{w}_1 = \sqrt{2}, \quad \tilde{w}_1^+ = 2ch\theta + \sqrt{4sh^2\theta + 2}, \quad \tilde{w}_1^- = 2ch\theta - \sqrt{4sh^2\theta + 2}, \quad (36)$$

and when $k_1 = k, k_2 = 0$,

$$\bar{w}_1 = \sqrt{4sh^2\theta + 2}, \quad \tilde{w}_1^+ = 2ch\theta + \sqrt{2}, \quad \tilde{w}_1^- = 2ch\theta - \sqrt{2}. \quad (37)$$

The result for Type III diagram can be obtained by inserting above data into Eq.(12):

$$K_2(\theta)^{(III)} = \frac{1}{8sh\theta} \left(\frac{-i}{8\sqrt{2}(\sqrt{2}ch\theta - 1)} + \frac{i}{8\sqrt{2}(\sqrt{2}ch\theta + 1)} - \frac{i}{4\sqrt{2}\sqrt{2sh^2\theta + 1}} \right) \times (-\beta^2) \times 4. \quad (38)$$

Adding the above contributions as well as the classical value 1, boundary reflection matrix for the heavy particle is given by

$$K_2(\theta) = 1 + \frac{i\beta^2}{16} \left(\frac{sh\theta}{ch\theta} - \frac{sh\theta}{ch\theta - 1} - \frac{sh\theta}{ch\theta - 1/\sqrt{2}} + \frac{sh\theta}{ch\theta + 1/\sqrt{2}} \right) + O(\beta^4). \quad (39)$$

The unwanted non-meromorphic terms exactly cancel out once again.

On the other hand, there are two ‘minimal’ boundary reflection matrices are known for $a_3^{(1)}$ model[3, 5]. None of these agrees with the perturbative result.

We have checked that this boundary reflection matrix at one loop order by perturbation theory satisfies the boundary crossing-unitarity relations as well as the boundary bootstrap equations:

$$\begin{aligned} K_1(\theta) K_1(\theta - i\pi) &= S_{11}(2\theta), \quad K_2(\theta) K_2(\theta - i\pi) = S_{22}(2\theta), \\ K_2(\theta) &= K_1(\theta + i\pi/4) K_1(\theta - i\pi/4) S_{11}(2\theta), \\ K_1(\theta) &= K_3(\theta). \end{aligned} \quad (40)$$

In one loop checks, the following identity is useful:

$$\frac{(x + B/2)}{(x)} = 1 + \frac{i\pi B}{2h} \frac{sh\theta}{ch\theta - \cos(x\pi/h)} + O(B^2). \quad (41)$$

The exact boundary reflection matrix is determined uniquely if we assume the strong-weak coupling ‘duality’:

$$\begin{aligned} K_1(\theta) &= [1/2][3/2][5/2], \\ K_2(\theta) &= [1/2][3/2]^2[5/2]. \end{aligned} \quad (42)$$

On the other hand, the most general solution can be written in the following form under the assumption of the strong-weak coupling ‘duality’:

$$K_1(\theta) = [1/2]^{a_1} [3/2]^{b_1} [5/2]^{c_1} [7/2]^{d_1}, \quad (43)$$

$$K_2(\theta) = [1/2]^{a_2} [3/2]^{b_2} [5/2]^{c_2} [7/2]^{d_2}.$$

Inserting the above into the boundary bootstrap equations, we can obtain linear algebraic relations among the exponents. Solving this system of equations yields

$$\begin{aligned} a_1 &= \textit{free}, & b_1 &= \textit{free}, & c_1 &= b_1, & d_1 &= a_1 - 1, \\ a_2 &= -a_1 + b_1 + 1, & b_2 &= a_1 + b_1, & c_2 &= a_1 + b_1 - 1, & d_2 &= -a_1 + b_1. \end{aligned} \quad (44)$$

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